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Control of the rolling of a wheel by changing its imbalance☆

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Abstract

The motion of a wheel (rolling without slipping) in a vertical (longitudinal) plane is considered. The load, which is modelled as a point mass, can be moved within the wheel in a hollow which is positioned along its diameter. By moving the load from one end of the hollow to the other in a specific way, it is possible to manage the rolling of the wheel. An algorithm is constructed for control of the position of the load using which the load is periodically moved from one end of the hollow to the opposite end while the wheel rolls (non-uniformly) over the surface.

The longitudinal motion of a single-wheeled device (a monocycle) can be managed somehow or other by moving bodies associated with it relative to the wheel. Monocycles have been described[1,2,1] which are set into motion using a pendulum mechanism mounted on them. Longitudinal motion of such devices occurs when there is a deviation of the pendulum (using a drive) from the stable lower equilibrium position or from the upper unstable equilibrium position. Such a device was already known in the nineteenth century.

The "Gyrowheel" monocycle,[3] constructed at the Institute of Mechanics of Moscow State University is a wheel equipped with a pendulum, by means of which longitudinal motion of the monocycle is achieved, and a gyroscopic stabilizer, which maintains its vertical position. The gyroscopic stabilizer also enables to change the direction of motion of the device. A monocycle has been described in Refs. 4,5 in which motion and stabilization are also ensured using a pendulum and a gyroscopic stabilizer. Some problems which touch on the stability and stabilization of the motion of a single-wheeled velocipede were considered in Ref. 6.

Below, we investigate the possibility of control of the rolling of a wheel by displacing a load along a hollow, located along a diameter, positioned within the wheel, in other words, by control of the imbalance of the wheel. Here, planar motion of the wheel without slipping and without loss of contact with the support is considered. When there is imbalance of the wheel, the vertical component of the reaction of the support may vanish, that is incompatible with the requirement that the wheel should roll without losing contact with the support. This must be taken into account when synthesizing the control if rolling of the wheel without loss of contact with the support is to be achieved.

When the device moves over an uneven surface, it has to surmount protuberances and cavities. When rising onto a protuberance and leaving a cavity, the device must surmount the slope. Because of this problem, the upward motion of the device along an inclined plane is considered here.

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[1] See also www.theriotwheel.com, www.jackiechabanias.com, www.americanroadshop.com, www.dself.dsl.pipex.com.

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The system studied in this paper models, in particular, the circus attraction in which an artist, located within a wheel sets it in motion by shifting its centre of mass in the plane of the wheel.

1. The mechanical model of the device

To investigate the motion of a single-wheeled device in the longitudinal plane, we will consider a mechanical system consisting of two absolutely rigid bodies: a wheel of mass M and radius R and a load of mass m, which we will simulate as a point mass (Fig. 1). Suppose the mass of the wheel is uniformly distributed throughout it and, consequently, its centre of mass is at the geometric centre O. The load is moved within the wheel in a hollow which is located along a diameter, changing the imbalance of the wheel. The distance of the load m from the point O is denoted by r. The hollow is symmetrical about the centre of the wheel and its length is equal to $2r_0$. The quantity r can therefore take values between $-r_0$ and r_0:

$$|r| \le r_0 \tag{1.1}$$

where, naturally, $r_0 \le R$.

Suppose that, at the beginning of the motion, the hollow in which the load m is located is orientated horizontally along the X axis. The angle of rotation of this hollow anticlockwise with respect to the horizon is denoted by φ and the displacement of the centre of mass of the wheel O along the X axis is denoted by x so that $\dot{x} = \dot{\varphi} R \cos\gamma$. Here, γ is the angle between the inclined plane and the horizontal plane (Fig. 1). If $\dot{\varphi} > 0$, then, when $\gamma > 0$, the wheel rolls upwards along the inclined plane.

The mechanical system being considered has two degrees of freedom, which are characterized by the generalized coordinates φ and r or x and r. The expressions for the horizontal component V_{Ox} and the vertical component V_{Ov} of the velocity vector, V_O, of the centre of the wheel O and the components V_{mx}, V_{my} of the velocity vector V_m of the point mass m have the form

$$\begin{aligned} V_{Ox} &= \dot{\varphi} R\cos\gamma, \quad V_{Oy} = \dot{\varphi} R\sin\gamma \\ V_{mx} &= \dot{\varphi}(R\cos\gamma - r\sin\varphi) + \dot{r}\cos\varphi, \quad V_{my} = \dot{\varphi}(R\sin\gamma - r\cos\varphi) - \dot{r}\sin\varphi \end{aligned} \tag{1.2}$$

Taking relations (1.2) into account, we obtain the following expression for determining the kinetic energy T of the system

$$2T = \{J + m[R^2 + r^2 - 2Rr\sin(\varphi+\gamma)]\}\dot{\varphi}^2 + m\dot{r}^2 + 2mR\dot{\varphi}\dot{r}\cos(\varphi+\gamma) \tag{1.3}$$

Here $J = 3MR^2/2$ is the moment of inertia of the wheel about a point on its rim.

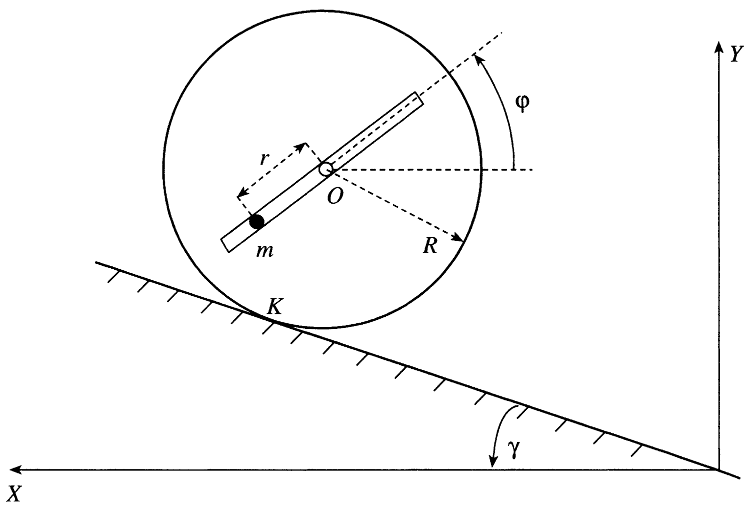

Fig. 1.

The potential energy of the system has the form

$$\Pi = (M+m)gR\varphi\sin\gamma - mgr\sin\varphi \tag{1.4}$$

Here $(M+m)gR\sin\gamma$ is the torque of the gravitational forces "tending" to roll the wheel downwards along an inclined plane at an angle γ to the horizon.

2. The equations of motion of the system

Using the Lagrange method of the second kind, it is possible, using expressions (1.3) and (1.4), to formulate the equations of motion of the system. The equation of motion corresponding to the generalized coordinate φ has the form

$$\begin{aligned} &\ddot{\varphi}\{J + m[R^2 + r^2 - 2Rr\sin(\varphi+\gamma)]\} + mR\ddot{r}\cos(\varphi+\gamma) = \\ &= m\{r[R\dot{\varphi}^2\cos(\varphi+\gamma) + g\cos\varphi] + 2\dot{\varphi}\dot{r}[R\sin(\varphi+\gamma) - r]\} - (M+m)gR\sin\gamma - \kappa\dot{\varphi} \end{aligned} \tag{2.1}$$

Here, $\kappa = \text{const}$ is the drag coefficient of the rotation of the wheel; it is assumed that the drag is proportional to the angular velocity $\dot{\varphi}$. Note that Eq. (2.1) can also be obtained[7] from the equation for the change in the angular momentum of the system about the instantaneous centre of velocities of the wheel K.

We will not write out the second equation of motion corresponding to the generalized coordinate r. Rather we shall subsequently assume that the distance r or, more accurately speaking, the acceleration $\ddot{r}$ or the velocity $\dot{r}$ is the control parameter in the system, and this second equation then only serves for determining the force by means of which the desired displacement of the load m within the hollow can be achieved.

We will now introduce the dimensionless time τ and dimensionless distance ρ according to the formulae

$$\tau = t\sqrt{g/R}, \quad \rho = r/R \tag{2.2}$$

The equation can then be written in dimensionless variables as follows

$$\begin{aligned} &\varphi''[j + 1 + \rho^2 - 2\rho\sin(\varphi+\gamma)] + \rho''\cos(\varphi+\gamma) = \\ &= \rho[\varphi'^2\cos(\varphi+\gamma) + \cos\varphi] + 2\varphi'\rho'[\sin(\varphi+\gamma) - \rho] - (1+\mu)\sin\gamma - \chi\varphi' \end{aligned} \tag{2.3}$$

Here, a prime denotes differentiation with respect to the dimensionless time τ and we have introduced the following dimensionless parameters of the system

$$\mu = \frac{M}{m}, \quad j = \frac{J}{mR^2} = \frac{3\mu}{2}, \quad \chi = \frac{\kappa}{m\sqrt{g}R^{3/2}} \tag{2.4}$$

In the dimensionless variables, inequality (1.1) takes the form

$$|\rho| \le \rho_0 \quad (\rho_0 = r_0/R \le 1) \tag{2.5}$$

Using the dimensionless variables (2.2) and the parameters (2.4), we can write expressions for the dimensionless horizontal R_x and vertical R_y components of the reaction of the support

$$R_x = (1+\mu)\varphi''\cos\gamma - (\varphi''\rho + 2\varphi'\rho')\sin\varphi - (\varphi'^2\rho - \rho'')\cos\varphi \tag{2.6}$$

$$R_y = (1+\mu)(1+\varphi''\sin\gamma) - (\varphi''\rho + 2\varphi'\rho')\cos\varphi - (\rho'' - \varphi'^2\rho)\sin\varphi \tag{2.7}$$

The dimensionless components of the reaction of the support R_x and R_y are obtained by dividing the dimensioned components by the weight of the load mg.

3. Statement of the problem

Suppose the wheel is located on a horizontal surface ($\gamma = 0$) in such a position that the hollow with the load is displaced from the vertical. If, at the same time, the load m is not located at the centre O, then the gravity force which is applied to it creates a torque which rotates the wheel in one or other direction. On account of this, by moving the load in a suitable manner within the hollow, it is possible to achieve motion of the wheel along a horizontal plane in the desired direction and even upwards along an inclined plane. In order to arrange for such motion, it is obviously necessary, at the time the load is underneath (when the hollow is vertical or almost vertical) to move it "rapidly" (using a drive mechanism) to the opposite end of the hollow which is located above. The load must be held at this end of the hollow until it once again drops down underneath during the rolling of the wheel. After this has happened, the load must again be moved to the opposite end of the hollow where it had previously been and so on.

In accordance with what has been said above, assuming for a start that the underlying surface is horizontal ($\gamma = 0$), we will consider the following algorithm for control the motion of the load, i.e. an algorithm for the change in the distance of the load ρ from the centre of the wheel O.

Suppose that, at the start of the motion when $\tau = 0$, the hollow in which the load is located is horizontal, the velocity of the wheel is zero and the load is (at rest) at the left-hand edge of the hollow, that is,

$$\varphi(0) = 0, \quad \varphi'(0) = 0, \quad \rho(0) = \rho_0, \quad \rho'(0) = 0 \tag{3.1}$$

Under these initial conditions, it is obvious that the wheel will begin to roll to the left when $\tau > 0$. Suppose the load further remains in the same position, that is, suppose $\rho'' = \rho' = 0$ and

$$\rho = \rho_0 \tag{3.2}$$

until the wheel has rotated through a quarter of a turn, that is, while

$$\varphi < \pi/2 \tag{3.3}$$

At a certain instant of time, τ_1, when the wheel has rotated by a quarter of a turn, that is, when

$$\varphi = \pi/2 \tag{3.4}$$

the load will be in the lowest position (closest to the support). At this instant of time, we begin to change the distance ρ or, more accurately speaking, the acceleration ρ'' as follows:

$$\rho'' = \begin{cases} -F_1 & \text{when} \quad \rho_1 \le \rho \le \rho_0 \\ 0 & \text{when} \quad -\rho_2 < \rho < \rho_1 \\ F_2 & \text{when} \quad -\rho_0 \le \rho \le -\rho_2 \end{cases} \tag{3.5}$$

where $F_1 > 0$, $F_2 > 0$, $\rho_1 > 0$ and $\rho_2 > 0$ are constant quantities. When $F_1 = F_2$ and $\rho_1 = \rho_2$, the graph of the function (3.5) becomes symmetrical about the origin of the coordinates. In accordance with expression (3.5), the load accelerates when $\rho_1 \le \rho \le \rho_0$, it moves along the hollow at a constant velocity when $-\rho_2 < \rho < \rho_1$ and slows down when $-\rho_0 \le \rho \le -\rho_2$.

We shall assume that the motion of the load within the hollow occurs during the time when the angle φ remains close to $\pi/2$. Then, if the constant F_2 is chosen to be too large, it follows from a consideration of expressions (2.3) and (2.7) that the vertical component R_y of the reaction of the support can vanish when $-\rho_0 < \rho < -\rho_2$ and subsequently becomes negative. In this case, contact between the wheel and the support can only be maintained using some kind of special device which restrains the wheel from "bouncing". On the other hand, if the constant F_1 is increased (when $\rho_1 \le \rho \le \rho_0$), the vertical component of the support reaction increases and there will be no loss of contact between the wheel and the support.

Relation (3.5) can be considered as a second-order differential equation in the variable ρ. There is no difficulty in writing out the solution of this equation. The velocity ρ', which is initially equal to zero (when $\rho = \rho_0$), decreases linearly with time in the interval $\rho_1 \le \rho \le \rho_0$ and, at the end of this interval when $\rho = \rho_1$

$$\rho' = -\beta = -\sqrt{2F_1(\rho_0 - \rho_1)} \tag{3.6}$$

The quantity ρ, being initially equal to ρ_0, also *decreases* according to a law which is quadratic in time in the case of the control (3.5). When $-\rho_2 < \rho < \rho_1$, the acceleration ρ'' is equal to zero and the velocity ρ' remains equal to the quantity (3.6). When $-\rho_0 \le \rho \le -\rho_2$, the velocity ρ', while remaining negative, increases linearly with time. It can be shown that, if the constants F_1, F_2, ρ_1, ρ_2 satisfy the condition

$$F_1(\rho_0 - \rho_1) = F_2(\rho_0 - \rho_2) \tag{3.7}$$

then the instant of time τ_2 approaches when the two equalities

$$\rho = -\rho_0, \quad \rho' = 0 \tag{3.8}$$

are at once satisfied. Condition (3.7) means that the first "pulse" in the control (3.5) is equal to the second.

By the time τ_2, the angle φ exceeds $\pi/2$, that is, $\varphi(\tau_2)>\pi/2$. Also, suppose $\varphi(\tau_2)<3\pi/2$. Then, by this instant of time, the load lies below its highest possible position and above its lowest position. We will assume that $\varphi'(\tau_2)\geq 0$; then, when $\tau>\tau_2$, we put $\rho''=\rho'=0$ and

$$\rho = -\rho_0 \tag{3.9}$$

In the case of the control (3.9), the torque of the gravity force applied to the wheel will continue to roll the wheel to the left (when $\gamma=0$).

At some instant of time τ_3, the wheel will have rotated by three quarters of a turn, that is, the following equality will hold

$$\varphi = 3\pi/2 \tag{3.10}$$

At this instant τ_3, the load will be in its lowest position, as in the case of condition (3.4) at the instant of time τ. Starting from the instant of time τ_3, we again change the distance ρ, by moving the load to the opposite end of the hollow. The law of variation of the acceleration ρ'' in this case only differs from (3.5) in the sign, since the distance ρ now has to *increase* from $-\rho_0$ to ρ_0.

We repeat the control cycle which has been described above when the angle φ reaches the values

$$\varphi = 5\pi/2, 7\pi/2, \ldots, \quad (2k+1)\pi/2, \quad k = 4, 5, \ldots \tag{3.11}$$

that is, for values of φ which are such that the difference $\varphi-\pi/2$ is a multiple of π.

If $F_1\to\infty$ and $\rho_1\to\rho_0$ and the product $F_1(\rho_0-\rho_1)$ remains constant such that $F_1(\rho_0-\rho_1)=\beta^2/2$, then the first of the "pulses" in the control (3.5) is "converted" into a delta-function of intensity $-\beta$ ($\beta=\text{const}>0$). In this case, the acceleration ρ'' is described by the formula

$$\rho'' = \begin{cases} -\beta\delta(\tau-\tau_1) & \text{when} \quad \rho = \rho \\ 0 & \text{when} \quad -\rho_2<\rho<\rho_0 \\ F_2 & \text{when} \quad -\rho_0\leq\rho\leq-\rho_2 \end{cases} \tag{3.12}$$

The vertical component of the reaction of the support becomes negative when $-\rho_0\leq\rho\leq-\rho_2$ if the magnitude of F_2 is sufficiently large, and hence we will not tend the quantity F_2 to infinity. If the magnitude of F_2 is not "too" large, then one can count on the fact that the vertical component of the reaction of the support will be positive during the whole time the wheel is rolling.

Under the control (3.12), the velocity ρ', which is equal to zero when $\rho=\rho_0$, then decreases stepwise to a value of $-\beta$, remains as such in the interval

$$-\rho_2<\rho<\rho_0 \tag{3.13}$$

and then increases linearly with time to zero.

So, at the instant of time τ_1 when condition (3.4) is satisfied, the control (3.12) can be "switched on" instead of the control (3.5). If $F_2(\rho_0-\rho_2)=\beta^2/2$, then the two equalities (3.8) are satisfied in the case of control (3.12) at the certain instant of time τ_2. After this time, the control is described by expression (3.9) until equality (3.10) is attained. After this equality has been reached at a certain instant of time τ_3, we "switch on" a control law which differs from (3.12) in sign.

It follows from a consideration of Eq. (2.3) that in the case of a jump $\rho'_+-\rho'_-=-\beta$ in the velocity ρ', the angular velocity φ' also undergoes a jump

$$\varphi'_+-\varphi'_- = \frac{\beta\cos(\varphi_*+\gamma)}{j+1+\rho_0^2-2\rho_0\sin(\varphi_*+\gamma)} \tag{3.14}$$

Formula (3.14) is written for the general case when the discontinuity occurs at an arbitrary value φ_* of the angle φ and an arbitrary angle of inclination γ of the underlying surface. If $\gamma = 0$ and $\varphi_* = \pi/2$, then the discontinuity (3.14) is equal to zero.

Below, in Section 6, the motion of a wheel which starts from a state of rest (3.1) and completes a certain periodic process obtained when $\rho = \text{const}$ is found numerically using the control algorithm (3.12) which has been described above.

Now suppose that, at the initial instant of time, the wheel is in a state of rest (3.1) on an inclined plane. It follows from a consideration of Eq. (2.3) that, in order for the wheel to be able to roll up the slope, it is *necessary* for the following condition to be satisfied

$$\rho_0 - (1+\mu)\sin\gamma > 0$$

We will rewrite this condition in the form

$$\sin\gamma < \frac{\rho_0}{1+\mu} \qquad \left(\sin\gamma < \frac{mr_0}{(M+m)R}\right) \tag{3.15}$$

Inequality (3.15) means that the vertical, dropped from the centre of mass of the system, intersects the inclined plane to the left of the point of contact K between the wheel and this plane (see Fig. 1). It is obvious that, when condition (3.15) is violated, the wheel cannot consistently roll up the slope whatever its initial velocity. Inequality (3.15) was obtained in Ref. 1 for the problem of control of the rolling of a wheel using a pendulum. In that problem, inequality (3.15) was both the necessary and sufficient condition for the wheel to move upwards along the inclined plane. In the problem being considered here, it is only a *necessary* condition.

4. The phase of the motion of the load along the hollow

Consideration of expression (2.7) as well as numerical investigations show that, under control (3.12) with large values of F_2, the vertical component R_y of the reaction of the support becomes negative at certain instants of time. However, in striving to obtain some kind of analytical relations to describe the motion of the mechanism and its properties, we shall neglect this fact and consider a control with two delta-functions instead of (3.12): when $\rho = \rho_0$ (as in (3.12)) and when $\rho = -\rho_0$. Under such a control, the velocity ρ' changes in accordance with the formula

$$\rho' = \begin{cases} 0 & \text{when} \quad \rho = \rho_0,\ \rho = -\rho_0 \\ -\beta & \text{when} \quad -\rho_0 < \rho < \rho_0 \end{cases} \tag{4.1}$$

According to expression (4.1), when $\rho = \rho_0$ the velocity ρ' changes stepwise from $\rho' = 0$ to $\rho' = -\beta$; subsequently, when

$$-\rho_0 < \rho < \rho_0 \tag{4.2}$$

the velocity $\rho' \equiv -\beta$ and, when $\rho = -\rho_0$, it changes stepwise in the opposite direction from $\rho' = -\beta$ to $\rho' = 0$. When the distance ρ becomes equal to $-\rho_0$, the angle φ changes and becomes equal to a certain value φ_{**}. Then, for the second discontinuity in the velocity ρ', which occurs when $\rho = -\rho_0$, we obtain the expression

$$\varphi'_+ - \varphi'_- = \frac{-\beta\cos(\varphi_{**}+\gamma)}{j+1+\rho_0^2+2\rho_0\sin(\varphi_{**}+\gamma)} \tag{4.3}$$

instead of relation (3.14).

Since under control (4.1) $\rho' \equiv -\beta$ and $\rho'' \equiv 0$ in the interval (4.2), the equation of motion (2.3) is somewhat simplified in this interval. As the independent variable, instead of the dimensionless time τ, we now consider the distance ρ, which decreases in a strictly monotonic manner from the value ρ_0 to the value $-\rho_0$. Then,

$$\frac{d\varphi}{d\tau} = -\frac{d\varphi}{d\rho}\beta, \quad \frac{d^2\varphi}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d\varphi}{d\tau}\right) = \frac{d}{d\tau}\left(-\frac{d\varphi}{d\rho}\beta\right) = \beta^2\frac{d^2\varphi}{d\rho^2} \tag{4.4}$$

and Eq. (2.3) (recall that now $\rho''=0$) takes the form

$$\begin{aligned}&\frac{d^2\varphi}{d\rho^2}[j+1+\rho^2-2\rho\sin(\varphi+\gamma)] = \\ &= \rho\left[\left(\frac{d\varphi}{d\rho}\right)^2\cos(\varphi+\gamma)+\frac{1}{\beta^2}\cos\varphi\right]+2\frac{d\varphi}{d\rho}[\sin(\varphi+\gamma)-\rho]-\frac{1+\mu}{\beta^2}\sin\gamma+\frac{\chi}{\beta}\frac{d\varphi}{d\rho}\end{aligned} \tag{4.5}$$

In accordance with condition (3.4), $\varphi(\rho_0)=\varphi(\tau_1)=\pi/2$. Using formula (3.14), we write the initial conditions for Eq. (4.5)

$$\varphi(\rho_0)=\frac{\pi}{2},\quad \frac{d\varphi}{d\rho}(\rho_0)=-\frac{1}{\beta}\frac{d\varphi}{d\tau}(\tau_1)-\frac{1}{\beta}(\varphi'_+-\varphi'_-)=-\frac{1}{\beta}\frac{d\varphi}{d\tau}(\tau_1)-\frac{\sin\gamma}{j+1+\rho_0^2-2\rho_0\cos\gamma} \tag{4.6}$$

If $\gamma=0$ (the underlying surface is horizontal), Eq. (4.5) is simplified and the second relation of (4.6) takes the form

$$\frac{d\varphi}{d\rho}(\rho_0) = -\frac{1}{\beta}\frac{d\varphi}{d\tau}(\tau_1) \tag{4.7}$$

In the case of a high modulus value of the velocity β, the time taken for the load to move from one end of the hollow to the other is short. By considering Eq. (4.5) when $\gamma=0$ and relation (4.7), it can be assumed that, in this case, the angle φ remains close to $\pi/2$ and that the angular velocity $d\varphi/d\rho$ changes by a finite amount. In order to show more rigorously that this is so, we will assume that the functions describing the change in the angle φ and in its derivative with respect to ρ can be represented in the form of series in negative powers of β

$$\varphi(\rho) = \frac{\pi}{2}+\frac{1}{\beta}\varphi_1(\rho)+\frac{1}{\beta^2}\varphi_2(\rho)+\cdots,\quad \frac{d\varphi}{d\rho} = \frac{1}{\beta}\frac{d\varphi_1}{d\rho}+\frac{1}{\beta^2}\frac{d\varphi_2}{d\rho}+\cdots \tag{4.8}$$

From expressions (4.7) and (4.8), we obtain

$$\frac{d\varphi_1}{d\rho}(\rho_0) = -\frac{d\varphi}{d\tau}(\tau_1) \tag{4.9}$$

We now substitute series (4.8) into Eq. (4.5) when $\gamma=0$. On collecting all terms containing β^{-1}, we obtain that the function $\varphi_1(\rho)$ satisfies the equation

$$\frac{d^2\varphi_1}{d\rho^2}[j+(1-\rho)^2] = 2\frac{d\varphi_1}{d\rho}(1-\rho)$$

the solution of which, when equality (4.9) is taken into account, can be represented in the form

$$\frac{d\varphi_1}{d\rho}(\rho) = \frac{d\varphi_1}{d\rho}(\rho_0)\frac{j+(1-\rho_0)^2}{1+(1-\rho)^2} = -\frac{d\varphi}{d\tau}(\tau_1)\frac{j+(1-\rho_0)^2}{j+(1-\rho)^2} \tag{4.10}$$

From this, we have

$$\frac{d\varphi_1}{d\rho}(-\rho_0) = -\frac{d\varphi}{d\tau}(\tau_1)\theta,\quad \theta = \frac{j+(1-\rho_0)^2}{j+(1+\rho_0)^2} \tag{4.11}$$

Note that, since $\theta<1$, the velocity of the wheel $\varphi'(\tau)$ by the time τ_2, when the load reaches the end of the hollow, turns out to be less than the velocity of the wheel at the initial instant of displacement of the load τ_1.

Integrating Eq. (4.10) and taking into account that $\varphi_1(\rho_0) = 0$, we obtain

$$\varphi_1(-\rho_0) = \varphi'(\tau_1)\frac{j+(1-\rho_0)^2}{\sqrt{j}}\left(\operatorname{arctg}\frac{1+\rho_0}{\sqrt{j}} - \operatorname{arctg}\frac{1-\rho_0}{\sqrt{j}}\right) = \varphi_1'(\tau_1)[j+(1-\rho_0)^2]\sigma$$

$$\sigma = \frac{1}{\sqrt{j}}\operatorname{arctg}\frac{2\rho_0\sqrt{j}}{j+1-\rho_0^2}$$

Hence, using the first of relations (4.8) and retaining the quantity of the order of β^{-1} in it, we have

$$\varphi_{**} = \varphi(-\rho_0) = \varphi(\tau_2) = \pi/2 + \beta^{-1}\varphi_1(-\rho_0) = \pi/2 + \beta^{-1}\varphi'(\tau_1)[j+(1-\rho_0)^2]\sigma \tag{4.12}$$

Substituting (4.12) into expression (4.3), we obtain that, when $\gamma = 0$ "immediately" after the instant when the distance ρ becomes equal to $-\rho_0$, the velocity φ' undergoes a jump of finite magnitude. For this discontinuity, neglecting terms of the order of β^{-1} and higher, we have the expression

$$\varphi'_+ - \varphi'_- = \varphi'(\tau_1)\theta\sigma$$

Thus, we obtain that the velocity φ', being at the right-hand end of the interval $-\rho_0 \le \rho \le \rho_0$, is equal to the quantity $\varphi'(\tau_1)$ and, in the case of a "large" value of β at the left-hand end of this interval (after the second discontinuity in the velocity ρ') becomes equal to

$$\varphi'(\tau_1)\theta + (\varphi'_+ - \varphi'_-) = \varphi'(\tau_1)\theta(1+\sigma) \tag{4.13}$$

When $\rho_0 = 0$, we have $\theta(1+\sigma) = 1$ and, when $\rho_0 > 0$, as can easily be shown, $\theta(1+\sigma) < 1$. Consequently, when $\rho_0 > 0$, the velocity of the wheel after the load moved from one end of the hollow to the other and then stopped (instantaneously) is found to be less than the velocity of the wheel before the load started to move.

Hence, approximate expressions (4.12) and (4.13) have been obtained, which describe the values of the angle φ and the angular velocity φ' at the instant of time τ_2 when conditions (3.8) are satisfied.

5. The rolling of a wheel in the case of a fixed position of the load

We will now find the velocity of the wheel at the instant of time τ_3 when condition (3.10) is satisfied. In the time interval

$$\tau_2 < \tau < \tau_3 \tag{5.1}$$

The position of the load is fixed: $\rho = -\rho_0$. It is therefore possible to use the energy integral to calculate the velocity $\varphi'(\tau_3)$, which holds when there is no rolling resistance, that is, when $\chi = 0$.

In dimensionless variables, the total energy of the system has the form

$$\begin{aligned}\frac{T+\Pi}{mgR} &= \frac{1}{2}[j+1+\rho^2 - 2\rho\sin(\varphi+\gamma)]\varphi'^2 + \frac{1}{2}\rho'^2 + \varphi'\rho'\cos(\varphi+\gamma) + \\ &+ (\mu+1)\varphi\sin\gamma - \rho\sin\varphi\end{aligned} \tag{5.2}$$

In expression (5.2), we put $\gamma = 0$ and also $\rho' = 0$ and $\rho = -\rho_0$. Then, neglecting terms of the order of β^{-1} and higher, we obtain the following expression for the constant energy integral h corresponding to the state (4.12), (4.13),

$$h = \frac{1}{2}[j+(1+\rho_0)^2][\varphi'(\tau_1)]^2\theta^2(1+\sigma)^2 + \rho_0 \tag{5.3}$$

Making the right-hand sides of (5.3) and (5.2) equal under condition (3.10), that is, when $\tau = \tau_3$, we obtain the relation between the velocities $\varphi'(\tau_3)$ and $\varphi'(\tau_1)$

$$[\varphi'(\tau_3)]^2 = [\varphi'(\tau_1)]^2\theta(1+\sigma)^2 + 4\eta \tag{5.4}$$

or

$$z_{n+1}^2 = \theta(1+\sigma)^2 z_n^2 + 4\eta \tag{5.5}$$

We have introduced the following notation here

$$z_n = \varphi'(\tau_1), \quad z_{n+1} = \varphi'(\tau_3), \quad \eta = \rho_0/[j+(1-\rho_0)^2]$$

Relation (5.4) or (5.5) describes the point image of the velocity z_n of the wheel at the instant when the load is at its lowest position to the velocity of the wheel z_{n+1} at the instant when the load is at the lowest position the next time, that is, after a half rotation of the wheel.If

$$\theta(1+\sigma)^2 < 1 \qquad \left(\frac{1}{\sqrt{j}}\operatorname{arctg}\frac{2\rho_0\sqrt{j}}{j+1-\rho_0^2} < \sqrt{\frac{j+(1+\rho_0)^2}{j+(1-\rho_0)^2}} - 1\right) \tag{5.6}$$

then Eq. (5.5) has the solution

$$z_n = z_{n+1} = z; \quad z = 2\sqrt{\eta/[1-\theta(1+\sigma)^2]} \tag{5.7}$$

We cannot prove in general form that inequality (5.6) holds. However, values of j and ρ_0 for which this condition would be violated have not been found in either approximate analytical or in numerical investigations.

Hence, control (4.1) in the case of a sufficiently high velocity of motion of the load β ensures periodic motion. The velocity at which the wheel rolls in these steady states, which corresponds to solution (5.7), will not, of course, be constant.

A graph of the function (5.5), constructed in the z_n, z_{n+1} plane for the following values of the parameters

$$j = 15 \ (M = 10 \text{ kg}, m = 1 \text{ kg}, R = 1 \text{ m}), \quad \rho_0 = 0.75 \ (r_0 = 0.75 \text{ m}) \tag{5.8}$$

is close to the bisectrix of the first quadrant and is not shown here. Intersection with the bisectrix occurs at the point (5.7): $z = z_n = z_{n+1} = 7.15$ (at the instant when the load is underneath, the angular velocity $\dot{\varphi} = 22.4 \text{ s}^{-1}$). It follows from inequality (5.6) that the slope of this graph to the abscissa is less than $\pi/4$ everywhere including at the point (5.7), which is indicative of the stability of the steady state of motion (5.7). However, since the slope of the graph of the function (5.5) is close to $\pi/4$ everywhere, the transition process slowly converges to a steady (periodic) process. An analysis of equality (5.7) shows that, when $\rho_0 \to 0$, the velocity $z \to \infty$. Numerical investigations of this relation, carried out for $j = 15$, indicate that the velocity z reaches a minimum when $\rho_0 = 0.67$.

Results are obtained in numerical investigations of the rolling of a wheel with control law (4.1) which are close to those described by the analytical relations presented above. These investigations show that periodic motions exist, the velocity φ' at the instant when the load is underneath is close to that calculated using formula (5.7) and that the periodic process is stable but the transition process converges to it slowly. The results of numerical investigations with control law (3.12), which are described below, are found to be similar from a qualitative point of view and close from a quantitative point of view.

6. Numerical investigations

The process of the acceleration of the wheel (when $\gamma = 0$ and $\chi = 0$) in (dimensionless) time, beginning from a state of rest, is shown in Fig. 2. The hollow in which the load is located is horizontal at the initial instant of time and the load is arranged (rests) at its left-hand edge. In other words, the initial conditions are described by equalities (3.1). The wheel parameters are given in relations (5.8). The control during the change in the position of the load is described by formula (3.12) in which

$$\beta = 10 \text{ms}^{-1}\sqrt{Rg} = 3.19, \quad \rho_0 = 0.75, \quad \rho_0 - \rho_2 = 0.9, \quad F_2 = \beta^2/2(\rho_0 - \rho_2) = 55.6$$

In accordance with the algorithm described in Section 3, the distance $\rho = \rho_0$ or $\rho = -\rho_0$ during the rest of the time.

It can be seen from Fig. 2 that the wheel velocity φ', being positive all the time, increases in the time interval $0 < \tau < .40$ but non-monotonically. The velocity increases monotonically when the load is located at one end of the

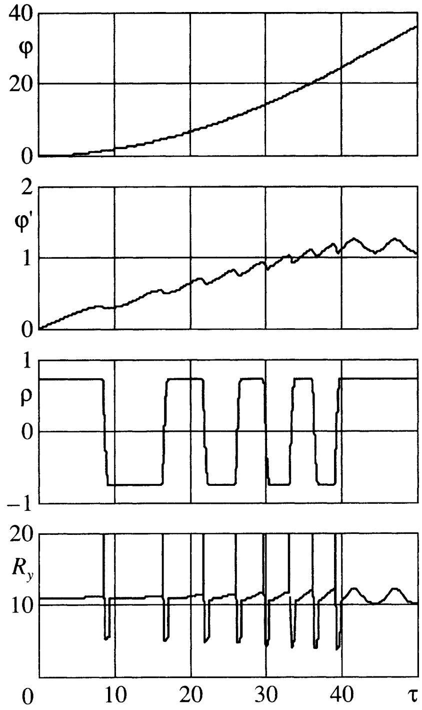

Fig. 2.

hollow and decreases as the load moves from one end of the hollow to the other, which corresponds to the theory developed in Section 4. Starting from the instant $\tau = 40$, the distance ρ, contrary to control law (3.12), "compulsorily" takes an equal constant value ρ_0. The velocity φ', starting from this instant, becomes a periodic function of time (see Fig. 2). In this case, the mean velocity of motion of the wheel is approximately 3.6 m/s. (If, at a certain instant of time, the load m is moved to the centre of the wheel O, its motion velocity becomes constant).

At the instant when the velocity of the load, which is located below, instantaneously changes from zero to a value of $-\beta$ or β, the vertical component R_y of the reaction of the support takes an infinitely large positive value. It is, of course, impossible to show this value on the graph. The change in the reaction R_y is shown (in dimensionless units) within the limits of 0 to 20 in Fig. 2. The value of the reaction R_y, which is a minimum during the rotation of the wheel through an angle π, is reached during the retardation of the load when it is close to the upper end of the hollow. It follows from Fig. 2 that this value decreases as the wheel accelerates. The minimum value of the vertical component R_y of the reaction of the support as the wheel accelerates, which is shown in Fig. 2, is reached on the last segment of the deceleration of the load and is 34.7 N for a weight of the whole mechanism equal to 107.9 N. For $40 < \tau < 50$ when $\rho = \rho_0$, the value of the reaction R_y oscillates about a value which is equal to the weight of the mechanism. The existence of upwardly directed "peaks", which arise during the acceleration of the load, and "troughs", which arise during the deceleration of the load, is characteristic of the graph for the change in the reaction R_y with time. These troughs become deeper as the velocity of the wheel increases and, between the peaks and troughs, the values of the reaction R_y are close to the weight of the mechanism.

By calculating the modulus of the ratio of the horizontal component (2.6) and vertical component (2.7) of the reaction of the support, it is possible to find the value of the friction coefficient of the wheel on a surface on which it can roll without slipping. For the motion shown in Fig. 2, this value is equal to 0.63. Such a friction coefficient is required in the final interval of the deceleration of the load.

The same is shown in Fig. 3 as in Fig. 2, that is, the process of the acceleration of the wheel but in the plane of the variables φ and φ'. Here, a series of curves, one arranged above the other, is shown in the range $0 \le \varphi \le 2\pi$. In the first curve, the very lowest, the angle φ actually changes from 0 to 2π. The second curve, which is located above the first curve, in fact corresponds to a change in the angle φ from 2π to 4π. The following third curve corresponds to a

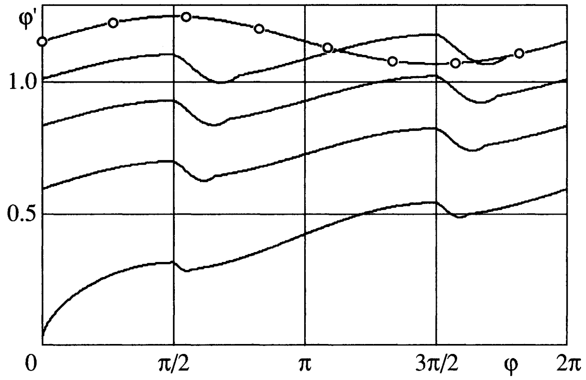

Fig. 3.

change in the angle φ from 4π to 6π, and so on. For the n-th curve, the angle φ actually changes from $2\pi(n-1)$ to $2\pi n$. The uppermost curve in Fig. 3 with the markers corresponds to the control $\rho = \rho_0$. In the case of this control, the imbalance of the wheel does not change and, when there is no resistance to motion, it rolls without deceleration in a periodic state. If the strip $0 \leq \varphi \leq 2\pi$ shown in Fig. 3 is rolled into a cylinder by superposing the lines $\varphi = 0$ and $\varphi = 2\pi$, the set of curves shown in the figure combines into a continuous smooth curve which is wound onto the cylinder. At the same time, the last curve (with the markers) is converted into a closed curve.

Of course, when there is resistance to the rolling of the wheel ($\chi \neq 0$), the acceleration of the wheel is lower.

In order to increase the mean velocity of the wheel in a periodic motion, the control algorithm described in Section 3 has to be used for a longer time, but only until the vertical component R_y of the reaction of the support becomes equal to zero. If the distance ρ consistently changes in accordance with this control algorithm, then, at certain instant of time (prior to reaching periodic state), the reaction of the support R_y vanishes, that is, the wheel bounces at this instant.

If it is assumed that the wheel is maintained on the support in some manner or other, then, under the control algorithm described in Section 3, the motion slowly reaches periodic state, which corresponds to the results obtained in Sections 4 and 5.

The control law which has been constructed is such that the load begins to move upwards along the hollow at the instant when it is at the bottom, by conditions (3.4), (3.10) and (3.11). If the motion of the load starts somewhat earlier, then, as numerical investigations have shown, the process is not substantially changed. Only the minimum value of the vertical component of the reaction of the support is increased somewhat.

Numerical investigations show that the maximum (critical) angle γ, for which the wheel, starting from a state of rest, can be raised upwards along an inclined plane in the case of the algorithm which has been constructed, is approximately equal to 2.1°. (In the case of the values of the parameters (5.8), inequality (3.15) takes the form $\gamma < 3.91^\circ$.) If the initial velocity of the wheel is not equal to zero, then, as numerical investigations show, it can surmount a steeper slope. Naturally, in the steady state of motion the velocity of the wheel decreases as the slope γ increases and, if the angle γ is close to the critical angle, the vertical component of the reaction of the support remains positive during the whole motion including under periodic state.

7. Conclusion

An algorithm for the control of the position of the load has been constructed under which the wheel, which is initially in a state of rest, accelerates and rolls with a constant mean velocity. If the velocity of the wheel is not bounded, then, under this control algorithm, the vertical component of the reaction of the support vanishes at a certain instant of time and the wheel bounces.

The motion of the wheel is non-uniform. By reducing the range of motion of the load within the wheel, it is possible to make this motion more uniform. At the same time, however, the time taken for the wheel to accelerate from a state of rest is increased. When there are two or more hollows within a wheel with a load in each of them, it is also possible to ensure a more uniform motion of the wheel using the control algorithm described above for each of the loads.

The control algorithm which has been constructed enables the wheel to travel up an inclined plane.

In order to slow down the rotating wheel, the motion of the load must occur in the opposite way to that described above for its motion when the wheel is to be accelerated. At the instant when the load is above (furthest away from the support), it is necessary to start to move the load towards the opposite end of the hollow which, at this instant, is at the bottom. The load must be kept at this end of the hollow until it is again raised to the top during the rolling of the wheel since, prior to being raised upwards, the load creates a torque which decelerates the wheel. Once the load is at the top, it must once again be moved to the opposite end of the hollow where it was located earlier, and so on.

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